Numerical Simulation of Maxwell Equation Using Finite Difference Method in Case of Subsurface Conductivity Parameter

Adam Sukma Putra, Sismanto

Department of Physics, Faculty of Mathematics and Natural Science, Universitas Gadjah Mada, Yogyakarta, Indonesia

*Correspondence Email: adamsukmaputra@ugm.ac.id

INFO ARTIKEL
Diterima : 02 Agustus 2022
Direvisi : 07 September 2022
Dipublikasikan : 12 September 2022

ABSTRACT
The aim of this project is to solve the Maxwell’s Equation using the finite difference (FD) method. We test the governing equation by discretizing the equation in 1-Dimensional System to describe the distribution of the magnitude of Electric and Magnetic Field within the subsurface layer. We assume that earth layer has constant conductivity ($\sigma$) where $\sigma >> \varepsilon \omega$ as a consequence that the system is homogenous and isotropic. We consider that the distribution of the field is described by the diffusion equation. We apply the modified form of FD method with Crank-Nicholson to improve the precision of the simulation.

Keywords: Electromagnetic, Maxwell Equation, Finite Difference, numerical simulation

1. Introduction

A study of Numerical Electromagnetics must rely on a firm base of knowledge in the foundations of electromagnetics as stated in Maxwell's equations. James Clerk Maxwell formulated Maxwell's equations in 1873 (Umashankar & Taflove, 1995).

\begin{align}
\nabla \cdot \mathbf{D} &= \rho \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
\n\nabla \cdot \mathbf{H} &= 0
\end{align}

where $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$

All classical electromagnetic phenomena are governed by a compact and elegant set of fundamental rules known as Maxwell’s equations. This set of four coupled partial differential equations was put forth as the complete classical theory of electromagnetics in a series of brilliant papers written by James Clerk Maxwell between 1856 and 1865, culminating in his classic paper (Umran & Robert, 2011). According to the equation, A most fundamental prediction of this theoretical framework is the existence of electromagnetic waves, a conclusion to which Maxwell arrived in the absence of experimental evidence that such waves can exist and propagate through empty space which is proved by Heinrich Hertz in 1887.
1.1 The Validity of Maxwell's Equations

The Maxwell's Equation is based on their consistency in all of experimental researches concerning in classical electromagnetic phenomena. It has the physical meaning described by the law based on the experimental fact. (Umran & Robert, 2011).

1. Faraday’s law shows that time-changing magnetic flux induces electromagnetic force

\[
\oint_C \mathbf{E} \cdot d\mathbf{l} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}
\]

The line integration over the contour \(C\) (direction of \(d\mathbf{l}\)) must be consistent with the direction of the surface vector \(d\mathbf{s}\) in accordance with the right-hand rule. Therefore

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)
\]

2. Gauss's law describes that the electric charges attract or repel one another with a force inversely proportional to the square of the distance between them (Coulomb's law)

\[
\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv
\]

Where the surface \(S\) enclose the volume \(V\). \(\rho\) is the volume charge density.

\[
\nabla \cdot \mathbf{D} = \rho \quad (2)
\]

3. Ampere’s law states that the line integral of the magnetic field over any closed contour must equal the total current.

\[
\oint_S \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot ds + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot ds
\]

where the contour \(C\) is that which encloses the surface \(S\), and \(\mathbf{J}\) is the electrical current density. It shows that timevarying electric fields produce magnetic fields.

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (3)
\]

4. Maxwell’s fourth equation is based on the fact that there are no magnetic charges (magnetic monopole). Therefore, magnetic lines always close on themselves.

\[
\oint_S \mathbf{B} \cdot ds = 0 \quad , \quad \nabla \cdot \mathbf{B} = 0 \quad (4)
\]

where the surface \(S\) encloses the volume \(V\). This equation can be derived from Biot-Savart law.

5. The continuity equation, which expresses the principle of conservation of charge in differential form.
\[ -\oint_S \mathbf{J} \cdot d\mathbf{s} = \frac{\partial}{\partial t} \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \]

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (5) \]

where the surface \( S \) encloses the volume \( V \).

In this research, we will perform the Finite difference methods which is widely used by physicist and geophysicist to perform simulation for electromagnetics waves. For example, Putra et all in 2019, by using the finite difference method (FD) approaching with the Crank-Nicholson (CN) scheme, they successfully provided the simulation of Finite Difference Time Domain of the Electromagnetics Waves. Another example of application of FD for FDTD has been published by skibachevskii A. (2016) in the bordered layer with absorbed boundary condition. Rump et al (2014) also published a paper in 2014 showing the numerical simulation of FD for scattered electromagnetics waves. According to this references, we tend to develop and demonstrate the FD method for the basic Maxwell equation to observe its characters as a conductive medium in earth layers.

2. Methodology

In this chapter, we will discuss both the formulation of Electromagnetic (EM) wave equation and its discretization using Finite difference method.

2.1. EM Wave Equation

The “constitutive relations” relate the electric field intensity \( \mathbf{E} \) to the electric flux density \( \mathbf{D} \) and similarly the magnetic field intensity \( \mathbf{B} \) to the magnetic flux density \( \mathbf{H} \). (Tjia, 2002)

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad (6) \]
\[ \mathbf{B} = \mu \mathbf{H} \quad (7) \]

Where \( \varepsilon \) is the permittivity and \( \mu \) is the permeability of the homogenous material. The relationship between the vacuum is

\[ \varepsilon = \varepsilon_r \varepsilon_0 \quad (8) \]
\[ \mu = \mu_r \mu_0 \quad (9) \]

where \( \varepsilon_r \) is relative permittivity and \( \mu_r \) is relative permeability \( \varepsilon_0 \) and \( \mu_0 \) are the vacuum constant \( (\varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{N}^{-1} \text{m}^{-2} \text{ and } \mu_0 = 4\pi \times 10^{-7} \text{ H/m}) \)

let see the Maxwell's equation (2).

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

We operate this equation with \( \nabla \times \) cross product (\( \nabla \times \mathbf{E} \)). the equation (2) become

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left( -\frac{\partial \mathbf{B}}{\partial t} \right) \]
\[ \nabla \times (\nabla \times \mathbf{E}) = \left( -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} \right) \quad (10) \]

Remember that

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (11) \]

And, for system without source

\[ \nabla \cdot \mathbf{E} = 0 \quad (12) \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (13) \]

By substituting the equation (11), (12) & (13) to the (10), therefore

\[ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} \]
\[ -\nabla^2 \mathbf{E} = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} \]
\[ -\nabla^2 \mathbf{E} = -\frac{\partial (\nabla \times \mu \mathbf{H})}{\partial t} \]
\[ -\nabla^2 \mathbf{E} = -\mu \frac{\partial (\nabla \times \mathbf{H})}{\partial t} \]
\[ -\nabla^2 \mathbf{E} = -\mu \frac{\partial (1 + \epsilon \frac{\partial \mathbf{E}}{\partial t})}{\partial t} \]

\[ \mathbf{J} = \sigma \mathbf{E}, \]

\[ \nabla^2 \mathbf{E} = \mu \frac{\partial}{\partial t} \left( \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \]
\[ \nabla^2 \mathbf{E} = \sigma \mu \frac{\partial \mathbf{E}}{\partial t} + \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (14) \]

Then from the Maxwell equation (3)

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

By giving the same treatment,

\[ \nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J} + \frac{\partial (\nabla \times \mathbf{D})}{\partial t} \]
\[ \nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \sigma (\nabla \times \mathbf{E}) + \epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \]

Substitute \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{B} = \mu \mathbf{H} \)

\[ \nabla^2 \mathbf{H} = \sigma \mu \frac{\partial \mathbf{H}}{\partial t} + \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad (15) \]

Finally we get two set equation of EM wave.
\[ \nabla^2 E = \sigma \mu \frac{\partial E}{\partial t} + \epsilon \mu \frac{\partial^2 E}{\partial t^2} \]  
(14)

\[ \nabla^2 H = \sigma \mu \frac{\partial H}{\partial t} + \epsilon \mu \frac{\partial^2 H}{\partial t^2} \]  
(15)

According to the subsurface earth layer that we treat them as the large conductor system, where the \( \sigma \gg \omega \epsilon \), therefore we assume that the \( \sigma \mu \frac{\partial H}{\partial t} \) is totally larger than \( \epsilon \mu \frac{\partial^2 H}{\partial t^2} \). As a result, the equation (14) and (15) can be simplified as:

\[ \nabla^2 E = \sigma \mu \frac{\partial E}{\partial t} \]  
(16)

\[ \nabla^2 H = \sigma \mu \frac{\partial H}{\partial t} \]  
(17)

The equation (16) and (17) is the induction process of the magnitude of field which follow the diffusion phenomena (Diffusion Regime). (Sutarno, 2014)

The equation (16) & (17) are solved numerically using FN method in 1 dimensional in order to describe the field induction distributed within the earth layer. We assume that the model of the earth is homogenous and isotropic. Therefore, the conductivity of the earth are constant along the dimensional axis. The simulation is done Using MATLAB.

### 2.2. Discretization using FN method

The FD method is used to solve the equation by discretization the partial differential equation into a square block with a specific interval. (Desai, 1972).

\[ \frac{\partial E}{\partial t} = \frac{E_{i+1,j} - E_{i,j}}{\Delta t} \]  
(18)

\[ \frac{\partial H}{\partial t} = \frac{H_{i+1,j} - H_{i,j}}{\Delta t} \]  
(19)

We use the central difference For 1-D system that the E and H change by depth (z direction), the equation become

\[ \nabla^2 E = \frac{\partial^2 E}{\partial z^2} = \frac{E_{i+1,j} - 2E_{i,j} + E_{i-1,j}}{\Delta z^2} \]  
(20)

\[ \nabla^2 H = \frac{\partial^2 H}{\partial z^2} = \frac{H_{i+1,j} - 2H_{i,j} + H_{i-1,j}}{\Delta z^2} \]  
(21)

By substituting, the EM equation become,

\[ \frac{E_{i,j+1} - E_{i,j}}{\Delta t} = \frac{1}{\sigma \mu} \left[ E_{i+1,j} - 2E_{i,j} + E_{i-1,j} \right] \]  
(22)

\[ \frac{H_{i,j+1} - H_{i,j}}{\Delta t} = \frac{1}{\sigma \mu} \left[ H_{i+1,j} - 2H_{i,j} + H_{i-1,j} \right] \]  
(23)

In order to increase the precision, we apply the Crank-Nicholson numerical method which has high numerical precision by averaging the central difference method. The equation (22) and (23) become

\[ \frac{E_{i,j+1} - E_{i,j}}{\Delta t} = \alpha \left[ \frac{(E_{i+1,j+1} - 2E_{i,j+1} + 2E_{i-1,j+1}) + (E_{i+1,j} - 2E_{i,j} + 2E_{i-1,j})}{\Delta z^2} \right] \]  
(24)
\[ \frac{H_{i,j+1} - H_{i,j}}{\Delta t} = \alpha \left[ \frac{(H_{i+1,j+1} - 2H_{i,j+1} + 2H_{i-1,j+1})}{\Delta z^2} \right] \] \hspace{1cm} (25)

Where \( \alpha = \frac{1}{2} \sigma \mu \)

![Crank-Nicholson Scheme](image)

Figure 1. Crank-Nicholson Scheme. (Bergara, 2009)

By separating for each variable, then we get

\[ \alpha E_{i,j+1} + (2 - 2\alpha)E_{i,j} + \alpha E_{i+1,j+1} = -\alpha E_{i-1,j+1} + (2 + 2\alpha)E_{i,j+1} - \alpha E_{i+1,j+1} \] \hspace{1cm} (26)

\[ \alpha H_{i,j+1} + (2 - 2\alpha)H_{i,j} + \alpha H_{i+1,j+1} = -\alpha H_{i-1,j+1} + (2 + 2\alpha)H_{i,j+1} - \alpha H_{i+1,j+1} \] \hspace{1cm} (27)

With the stability. (Bergara, 2009)

\[ \eta(k) = \frac{1 - \sin^2 \left( \frac{k\Delta x}{2} \right)}{1 + \sin^2 \left( \frac{k\Delta x}{2} \right)} \leq 1 \]

The equation (26) and (27) can be represented in matrices

For \( E \)

\[
\begin{bmatrix}
2 + 2\alpha & -\alpha & 0 \\
-\alpha & 2 + 2\alpha & \cdots \\
0 & \cdots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
E_{1,j+1} \\
\vdots \\
E_{n,j+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
-\alpha E_{0,j+1} \\
\vdots \\
-\alpha E_{n,j+1} \\
\end{bmatrix}
\]

For \( H \)

\[
\begin{bmatrix}
2 + 2\alpha & -\alpha & 0 \\
-\alpha & 2 + 2\alpha & \cdots \\
0 & \cdots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
H_{1,j+1} \\
\vdots \\
H_{n,j+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
-\alpha H_{0,j+1} \\
\vdots \\
-\alpha H_{n,j+1} \\
\end{bmatrix}
\]

Whichever of the boundary condition we have, the crank-Nicholson Scheme,

\[ M_{j+1}^L E_{j+1}^L = r_j^L r_j^L E_j + M_j^L E_j \] \hspace{1cm} (28)

\[ M_{j+1}^L H_{j+1}^L = r_j^L r_j^L H_j + H_j^L E_j \] \hspace{1cm} (29)

According to the exact solution, if the wave is a plane wave travelling in the z direction, the boundary condition for the wave equation given as (Sutarno, 2014)
\[ E(x, t) = E_0 e^{-at} \cos(\omega t - az) \]  
\[ H(x, t) = H_0 e^{-at} \cos(\omega t - az) \]

Where \( a = \sqrt{\frac{\omega \mu \sigma}{\varepsilon}} \)

For \( t=0, \ z=\infty \),
\[ E(z, 0) = E_0 e^{-az} \cos(az) \quad (32) \]
\[ H(z, 0) = H_0 e^{-az} \cos(az) \quad (33) \]

For \( t=\infty, \ z=0 \),
\[ E(0, t) = E_0 \cos(\omega t) \quad (34) \]
\[ H(0, t) = H_0 \cos(\omega t) \quad (35) \]

For \( t=\infty, \ z=\infty \),
\[ E(0, t) = E_0 \]
\[ H(0, t) = H_0 \quad (35) \]

3. Results and Discussion

According to the FD methods presented by putra et al (2014), they performed the neumann boundary condition as we applied it in this simulation. Here are the parameter which used in this simulation

<table>
<thead>
<tr>
<th>Physical Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>( 10^4 \text{Hz} )</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>( 0.01 \Omega^{-1} )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( 1.5 \times 4 \pi \times 10^{-7} \text{H/m} )</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>( 2 \times 8,85 \times 10^{-12} )</td>
</tr>
<tr>
<td>( E_0 )</td>
<td>1</td>
</tr>
<tr>
<td>( H_0 )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

We simulated both for \( E \) and \( H \) in \( z \) direction (depth).

![EM diffusion](image.png)

Figure 2. \( E_z \) and \( H_z \) vs time

We can see that with \( E_0=1 \) and \( H=0.5 \), the maximum value of the magnetic field is a half from the electric field. The value changes time by time as a consequence that there are an attenuation ini boundary condition represented by \( e^{-az} \). The initial condition has the sinusoidal component
\( \cos(az) \) and \( \cos(\omega t) \). It describe that the field source emit a sinusoidal wave along the distance. (see figure 3.)

By combining figure 2. And 3. We can see the behavior of the E and H changes time by time. (figure 4.)

When the value of \( E_0 \) is equal to \( H_0 \), the data is overlap. But in field work it is rarely existing. It is because the source of magnetic field is likely weaker than the electric field according to the Maxwell's equation. Naofumi N, in 2018 has been both solved the Maxwell's equation theoretically and numerically using the Eulerian FDTD methods. And the result seems to be matched with our numerical result which is solved the equation using the CN schemes. It is, therefore, the FD method can be widely applied for simulate the wave equation using variance of modification such as Eulerian, CN, and etc.

4. Conclusion

Maxwell's equation has been successfully describe all of the classical electrodynamic phenomena. The numerical modeling of the Maxwell’s equation has wide application for yielding any system that obtain physical parameter. For example, in this paper the Maxwell’s equation has been success on describing the wave propagation within subsurface layer. The crank-Nicholson method can be applied successfully for solving the equation numerically.

For further research, we can assume that, \( \sigma << \epsilon \omega \), therefore, the EM equation propagate as the function of wave within subsurface layer. Or for more accuracy and complexity, we can use both of the parameter (\( \sigma \) & \( \epsilon \)) then we solve it numerically. In numerical simulation, we may also applied another approach such as Leaf-Frog method, Rungge-Kutta or any proper schmes for the methods.
5. References


